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Free Split Bands

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Abstract: We solve the word problem for the free objects in the variety consisting of bands with a semilattice transversal. It follows that every free band can be embedded into a band with a semilattice transversal.

Keywords: Free band, Split band, Semilattice transversal

1 Introduction

We refer to[3] and[6] for a general background and as references to terminology used in this paper.

Recall that a *band* is a semigroup where every element is an idempotent. The *Green relation* $\mathcal{D}$ is the least semilattice congruence on a band, and so every band is a semilattice of its $\mathcal{D}$-classes; the $\mathcal{D}$-classes themselves form rectangular bands.[5] We shall be interested in bands S for which the least semilattice congruence $\mathcal{D}$ *splits*, that is, there exists a subsemilattice S° of S which intersects each $\mathcal{D}$-class in exactly one element. Such a subsemilattice S° of S will then be called a *semilattice transversal* of S.

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If the band S has a semilattice transversal S°, then we can associate to every $a \in S$ the unique $a^\circ \in S^\circ$ which is $\mathcal{D}$-related to a. The unary operation $S \mapsto S, a \mapsto a^\circ$ is an idempotent $\mathcal{D}$-class preserving endomorphism of S which induces the $\mathcal{D}$-relation on S, and in particular, S° is a *retract* of S. The *unary band* $((S,\cdot,^\circ))$ thus obtained obviously satisfies, apart from the associative law, the identities

$$x \approx x^2, \quad xx^\circ x \approx x, \quad x^\circ x x^\circ \approx x^\circ,$$
$$(xy)^\circ \approx x^\circ y^\circ \approx y^\circ x^\circ, \quad (x^\circ)^\circ \approx x^\circ.$$

(1)

One readily verifies that the last identity $(x^\circ)^\circ \approx x^\circ$ follows in fact from the previous ones. It is not difficult to see that if a unary band $((S,\cdot,^\circ))$ satisfies the identities (1), then $S^\circ = \{a^\circ | a \in S\}$ is a semilattice transversal of the band $(S,\cdot)$. For this reason we call the unary bands which satisfy the identities (1) *split bands*.

The variety of bands will be denoted by $\mathbf{B}$ and the variety of all split bands will be denoted $\mathbf{B}^\circ$. For a nonempty set X, $F_{\mathbf{B}}(X)$ and $F_{\mathbf{B}^\circ}(X)$ denote the free objects on X in $\mathbf{B}$ and $\mathbf{B}^\circ$, respectively. As the abstract indicates, the purpose of this paper is to solve the word problem for $F_{\mathbf{B}^\circ}(X)$ and to show that $F_{\mathbf{B}}(X)$ can be isomorphically embedded into the multiplicative reduct of $F_{\mathbf{B}^\circ}(X)$. The solution of the word problem for $F_{\mathbf{B}}(X)$ is well understood and the reader will find all the necessary details in Sect 4.5 of[3] where appropriate references to[1] and[2] are given. Our solution to the word problem for $F_{\mathbf{B}^\circ}(X)$ is similar though slightly more complicated. While $\mathbf{B}^\circ$ is, like $\mathbf{B}$, locally finite, for any finite nonempty set X the (finite) free object $F_{\mathbf{B}^\circ}(X)$ is much larger than $F_{\mathbf{B}}(X)$.

There is something more enigmatic about all this. It turns out that if a band S has a semilattice transversal S° then the union of all the $\mathcal{L}$-classes [$\mathcal{R}$-classes] in S of the elements of S° is a left [right] regular band which is a transversal of the $\mathcal{R}$-classes [$\mathcal{L}$-classes] of S. This follows from a very special application of Proposition 2.3 and Corollary 2.4 of[8] and below we intend to give a short independent proof of this fact in the special circumstances we consider here.

Result 1.1

Let $(S,\cdot,^\circ)$ be a split band with semilattice transversal $S^\circ = \{a^\circ | a \in S\}$. Then $I_S = \{aa^\circ | a \in S\}$ is a left [right] regular subband of $(S,\cdot)$ with semilattice transversal S°, and Λ_S is a transversal of the $\mathcal{R}$-classes [$\mathcal{L}$-classes] of $(S,\cdot)$.

Proof

Clearly, for any $a \in S$, the intersection of I_S and the $\mathcal{D}$-class D_a of a is the the $\mathcal{L}$-class L_{a° of a° and I_S intersects the $\mathcal{R}$-class R_a of a in the unique element aa°. It suffices to prove that I_S is a subsemigroup of S, or in particular, that for any $a, b \in S$we have that $(aa^\circ)(bb^\circ)\mathcal{L}(ab)^\circ$. In any case $a^\circ b^\circ = b^\circ a^\circ = (ab)^\circ$ is $\mathcal{D}$-related to $(aa^\circ)(bb^\circ)$ in S and since this $\mathcal{D}$-class is a rectangular band it follows that $(b^\circ a^\circ)(aa^\circ bb^\circ)\mathcal{L}(aa^\circ)(bb^\circ)$. Applying the identities (1) we find that

$$
\begin{aligned}
(b^\circ a^\circ)(aa^\circ bb^\circ) &= b^\circ(a^\circ aa^\circ)bb^\circ \\
&= b^\circ a^\circ bb^\circ \\
&= a^\circ b^\circ bb^\circ \\
&= a^\circ b^\circ = (ab)^\circ.
\end{aligned}
$$

Thus $(aa^\circ)(bb^\circ)\mathcal{L}(ab)^\circ$ as required. □

The I_S and Λ_S mentioned in Result 1.1 may well serve as a means to coordinatize S and one would expect that such a coordinatization would set the stage for a structure theorem of split bands in terms of the left and right regular bands I_S and Λ_S akin to, but simpler than the construction in II.1 of.[6] For a free split band $F = F_{\mathbf{B}^\circ}(X)$, it is easy to characterize the elements of the left [right] regular split band Λ_F (see Corollary 2.2 and Theorem 2.5). By left-right duality, I_F is anti-isomorphic to Λ_F.

The variety $\mathbf{RRB}^\circ$ $\mathbf{LRB}^\circ$ of right [left] regular split bands is the subvariety of $\mathbf{B}^\circ$ determined by the additional identity $xyx \approx yx$ $xyx \approx xy$. Thus, with the notation of Result 1.1, I_S and Λ_S belong to $\mathbf{LRB}^\circ$ and $\mathbf{RRB}^\circ$, respectively. As we shall see, if $F = F_{\mathbf{B}^\circ}(X)$ is a free split band, then I_S and Λ_S should not be assumed to be free on X in $\mathbf{LRB}^\circ$ and $\mathbf{RRB}^\circ$, respectively.

It is time to put our paper in the context of current research. The adequate terminology *split band* is not of our invention but already occurs in[4] where the authors give

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a structure theorem for orthodox semigroups for which the least inverse semigroup congruence splits. Theorem 2 of[9] gives a structure theorem for the members of $\mathbf{RRB}^\circ$ in the manner of Theorem II.1.6 of.[6] Combining this result of Yoshida with its dual and with Theorem 2 of,[7] one obtains a structure theorem for the members of $\mathbf{B}^\circ$. We would, however, like to draw the reader's attention to the all encompassing paper[8] which has already been mentioned above, and which in its Example 2.15 introduces a variety of unary semigroups (whose members are all regular semigroups) which contains $\mathbf{B}^\circ$as a subvariety.

2 Free split bands

In this section we give a solution of the word problem for the free object $F_{\mathbf{B}^\circ}(X)$ in the variety $\mathbf{B}^\circ$ on a nonempty set X of *variables*.

We let X° be a set disjoint of X and $X \mapsto X^\circ, x \mapsto x^\circ$, a bijection. The elements of $X \cup X^\circ$ will be called *letters*. The identity of the free monoid $(X \cup X^\circ)^*$ is the empty word 1, thus $(X \cup X^\circ)^* = (X \cup X^\circ)^+ \cup \{1\}$. For any $w \in (X \cup X^\circ)^*$ we define the *content* $c(w)$ of w inductively by

$$\begin{aligned} c(1) &= \emptyset, \\ c(x) &= c(x^\circ) = \{x\}, \qquad x \in X, \\ c(y_1 \cdots y_{n-1} y_n) &= c(y_1 \cdots y_{n-1}) \cup c(y_n), \\ & n > 1, y_1, \dots, y_n \in X \cup X^\circ. \end{aligned}$$

We let β be the congruence relation of the free semigroup $(X \cup X^\circ)^+$ generated by the pairs

$$\begin{gathered} (w, w^2), \qquad w \in (X \cup X^\circ)^+, \\ (x, xx^\circ x), (x^\circ, x^\circ x x^\circ), \qquad x \in X, \\ (x^\circ y^\circ, y^\circ x^\circ), \qquad x, y \in X. \end{gathered}$$

(2)

One readily verifies that $(X \cup X^\circ)/\beta$ is a band generated by the elements of the form $x\beta$ or $x^\circ\beta$, and Green's $\mathcal{D}$-relation on this band is given by

$$v\beta \, \mathcal{D} \, w\beta \Leftrightarrow c(v) = c(w).$$

(3)

Also, the element of the form $(x_1^\circ \dots x_n^\circ)\beta, n \geq 1, x_1, \dots, x_n \in X$, constitute a subsemilattice which intersects every $\mathcal{D}$-class exactly once. Given $w \in (X \cup X^\circ)^+$ with content $c(w) = \{x_1, \dots x_n\}$ we let $(w\beta)^\circ$ be the unique element $(x_1^\circ \cdots x_n^\circ)\beta$ of this semilattice which is $\mathcal{D}$-related to $w\beta$ in $(X \cup X^\circ)^+/\beta$: the mapping $w\beta \to (w\beta)^\circ$ yields an idempotent endomorphism which induces the $\mathcal{D}$-relation. The unary band thus obtained will be denoted by F.

For the sake of simplicity we drop the notation β and we use "$=$" to denote equality in F. Thus for $v, w \in (X \cup X^\circ)^+$ we write $v = w$ (in F) instead of $v\beta w$. We shall reserve "$\equiv$" for the equality in $(X \cup X^\circ)^*$. We shall denote the semilattice transversal of F consisting of the elements $x_1^\circ \dots x_n^\circ, n \geq 1, x_1, \dots x_n \in X$, by F°. Clearly F is a model of $F_{\mathbf{B}^\circ}(X)$: if $\iota: X \to F$, then for every $S \in \mathbf{B}^\circ$ and every mapping $\varphi: X \to S$, there exists a (unique) homomorphism of unary bands $\overline{\varphi}: F \to S$such that $\varphi = \iota\overline{\varphi}$. If X is finite then so is $X \cup X^\circ$, and therefore F is also finite, since finitely generated bands are finite. In other words, the variety $\mathbf{B}^\circ$ is *locally finite*. From this it follows that there exists an algorithm which decides whether for given $v, v, w \in (X \cup X^\circ)^+$ we have that $v = w$ in F. We shall give an algorithm which is transparent enough to be useful. The algorithm which we set out to describe is similar to the algorithm given in[2] for free bands.

We shall need some invariants. In our context, a property of a word which belongs to $(X \cup X^\circ)^+$ is called an *invariant* if whenever $v = w$ in F and v satisfies this property, then so does w. To "have the same content" is such an invariant: recall that for $v, v, w \in (X \cup X^\circ)^+, c(v) = c(w)$ if and only if $v\mathcal{D}w$ in F, or if and only if $v\mathcal{D}w$ in F.

For any $w \in (X \cup X^\circ)^*$ we define $w_L \in X^*$ to be the word obtained from w by deleting first every occurrence of an $x \in X$ in w when preceded somewhere in w by x or x°, and then deleting every occurrence of any $y^\circ \in X^\circ$. Thus for instance, for $x, y, z \in X$,

$$\begin{aligned} (x^\circ y^\circ x x^\circ z^\circ y z)_L &\equiv 1, \\ (x^\circ y x x^\circ y^\circ z y z^\circ)_L &\equiv yz. \end{aligned}$$

The letters of such words w_L (if any) belong to X and are necessarily distinct. If $w_L \not\equiv 1$, then the last letter of $w_L$will be denoted by $\bar{l}(w)$. We can then uniquely write $w \equiv l(w)\bar{l}(w)u$ for some prefix $l(w)$ of w where $\bar{l}(w) \notin c(l(w))$ and some suffix u. Here $l(w)$ or u may well be empty. Thus, for instance,

$$\begin{aligned}\bar{l}(x^\circ yxx^\circ y^\circ zyz^\circ) &= z\\ l(x^\circ yxx^\circ y^\circ zyz^\circ) &= x^\circ yxx^\circ y^\circ.\end{aligned}$$

If $w_L = 1$, it will be convenient to put $l(w) \equiv \bar{l}(w) \equiv 1$.

In a left-right dual way we define, for $w \in (X \cup X^\circ)^*$, the word $w_R \in X^*$ and, whenever $w_R \not\equiv 1$, the variable $\bar{r}(w)$ and the suffix $r(w)$ of w. Then if $w_R \not\equiv 1$, $w \equiv v\bar{r}(w)r(w)$ in $(X \cup X^\circ)^*$ for some prefix v. If $w_R \equiv 1$ we put $r(w) \equiv \bar{r}(w) \equiv 1$.

Lemma 2.1

For $w \in (X \cup X^\circ)^+$ with $c(w) = \{x_1, \dots, x_n\}$,

$$\begin{aligned}&w \quad \mathcal{R} w x_1^\circ \cdots x_n^\circ = x_1^\circ \cdots x_n^\circ \text{ in F if } w_L \equiv 1, and\\ &w \quad \mathcal{R} w x_1^\circ \cdots x_n^\circ = l(w)\bar{l}(w) x_1^\circ \cdots x_n^\circ \text{ in F otherwise.}\end{aligned}$$

Proof

If $w \in (X \cup X^\circ)^+$ with $c(w) = \{x_1, \dots, x_n\}$, then $x \in X^+$ or $w \equiv w_0 y_1^\circ w_1 \dots y_k^\circ w_k$ where $w_0, \dots, w_k \in X^*$ and $y_1, \dots y_k \in c(w)$ for some $k \geq 1$.

First we assume that $w_L \equiv 1$. Then w can be written as $w = y_1^\circ w_1 \cdots y_k^\circ w_k$, $k \geq 1$, where $c(w) = \{y_1, \dots, y_k\}$ such that, for every $1 \leq i \leq k$, we have that $c(w_i) \subseteq \{y_1, \dots, y_i\}$. We prove by induction that for all $1 \leq i \leq k$ we have that $w = y_1^\circ \cdots y_i^\circ w$ in F. This is obviously true for $i = 1$. Suppose that for $i < k$ we have that

$$w = y_1^\circ \cdots y_i^\circ w = y_1^\circ \cdots y_i^\circ y_1^\circ w_1 y_2^\circ w_2 \cdots y_i^\circ w_i y_{i+1}^\circ \cdots y_k^\circ w_k.$$

Since

$$c(y_1^\circ \cdots y_i^\circ y_1^\circ w_1 \cdots y_i^\circ w_i) = y_1, \dots, y_i = c(y_1^\circ \cdots y_i^\circ),$$

it follows that

$$y_1^\circ \cdots y_i^\circ \mathcal{R} y_1^\circ \cdots y_i^\circ y_1^\circ w_1 \cdots y_i^\circ w_i$$

in F. Therefore

$$\begin{aligned} y_1^\circ \cdots y_i^\circ y_1^\circ w_1 \cdots y_i^\circ w_i y_{i+1}^\circ &= y_{i+1}^\circ y_1^\circ \cdots y_i^\circ y_1^\circ w_1 \cdots y_i^\circ w_i y_{i+1}^\circ \\ &= y_1^\circ \cdots y_i^\circ y_{i+1}^\circ y_1^\circ w_1 \cdots y_i^\circ w_i y_{i+1}^\circ, \end{aligned}$$

since the elements of F which are $\mathcal{R}$-related to an element of the semilattice transversal F° of F form a right regular band by Result 1.1. It follows that $w = y_1^\circ \cdots y_i^\circ y_{i+1}^\circ w$. Using induction and the fact that $x_1^\circ \cdots x_n^\circ = y_1^\circ \cdots y_k^\circ$ we find that $w = x_1^\circ \cdots x_n^\circ w$. Since $w \mathcal{D} x_1^\circ \cdots x_n^\circ$ in F, we thus have that $w \mathcal{R} w x_1^\circ \cdots x_n^\circ = x_1^\circ \cdots x_n^\circ w x_1^\circ \cdots x_n^\circ = x_1^\circ \cdots x_n^\circ$.

We next consider the case where $w_L \not\equiv 1$. Then there exists $\bar{l}(w) \in X$ such that $w \equiv l(w)\bar{l}(w)u$ for some $l(w), u \in (X \cup X^\circ)^*$. If $c(w) = \{x_1, \cdots, x_n\}$, we may as well assume that $c(l(w)\bar{l}(w)) = \{x_1, \cdots, x_j\}$ for some $j \le n$. Then

$$\begin{aligned} w &= l(w)\bar{l}(w)(l(w)\bar{l}(w))^\circ l(w)\bar{l}(w)u \text{ (since } l(w)\bar{l}(w)\mathcal{D}(l(w)\bar{l}(w))^\circ \text{ in } F) \\ &= l(w)\bar{l}(w)x_1^\circ \cdots x_j^\circ l(w)\bar{l}(w)u \\ &= l(w)\bar{l}(w)v \end{aligned}$$

where $v \equiv x_1^\circ \cdots x_j^\circ l(w)\bar{l}(w)u$ is such that $v_L = 1$ and $c(v) = \{x_1, \dots, x_n\} = c(w)$. By the first part of the proof, $v = x_1^\circ \dots x_n^\circ v$ and thus $w = l(w)\bar{l}(w)x_1^\circ \cdots x_n^\circ v$. Since w and $l(w)\bar{l}(w)x_1^\circ \cdots x_n^\circ$ have the same content, they must then be $\mathcal{R}$-related in F. Further, $x_1^\circ \cdots x_n^\circ v x_1^\circ \cdots x_n^\circ = x_1^\circ \cdots x_n^\circ$ and so $w x_1^\circ \cdots x_n^\circ = l(w)\bar{l}(w)x_1^\circ \cdots x_n^\circ$. □

From Lemma 2.1 and its dual we have the following.

Corollary 2.2

Let $w \in (X \cup X^\circ)^+$, $c(w) = \{x_1, \dots, x_n\}$.

If $w_L \not\equiv 1 \not\equiv w_R$, then

$$w \quad \mathcal{R} l(w)\bar{l}(w)x_1^\circ \cdots x_n^\circ \mathcal{L} x_1^\circ \cdots x_n^\circ \mathcal{R} x_1^\circ \cdots x_n^\circ \bar{r}(w)r(w)\mathcal{L} w$$

and $w = l(w)\bar{l}(w)x_1^\circ \cdots x_n^\circ \bar{r}(w)r(w)$.

If $w_L \not\equiv 1 \equiv w_R$, then

$w = l(w)\bar{l}(w)x_1^\circ \cdots x_n^\circ \mathcal{L} x_1^\circ \cdots x_n^\circ$.

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If $w_L \equiv 1 \not\equiv w_R$, then

$w = x_1^\circ \cdots x_n^\circ \bar{r}(w) r(w) \mathcal{R} x_1^\circ \cdots x_n^\circ$.

If $w_L \equiv 1 \equiv w_R$, then $w = x_1^\circ \cdots x_n^\circ$.

That the $l, \bar{l}, r$ and $\bar{r}$ yield invariants follows from the following sequence of results.

Lemma 2.3

Let $v, w \in (X \cup X^\circ)^+$ be such that $v = w$ in F. Then $v_L \not\equiv 1$ if and only if $w_L \not\equiv 1$. If this is the case, then $\bar{l}(v) \equiv \bar{l}(w)$ and either $l(v) \equiv 1 \equiv l(w)$ or $l(v) = l(w)$ in F.

Proof

If $v = w$ in F then v can be transformed into w by using a finite sequence of transformations of the form $psq \to ptq$ where $p, q \in (X \cup X^\circ)^*$ and where (s, t) or (t, s) is one of the pairs listed in (2). It therefore suffices to prove the statement of the theorem for the special case where $v \equiv psq$ and $w \equiv ptq$ where (s, t) is any of the pairs listed in (2). The proof is not much harder than the corresponding verification for free bands as in the proof of Lemma 4.5.1 of.3 □

Using an inductive argument we thus have the following.

Corollary 2.4

If $v, w \in (X \cup X^\circ)^+$ are such that $v = w$ in F, then $v_L \equiv w_L$.

It will be convenient to denote by F^1 the unary band F with an identity 1 adjoined. The unary operation of F can be extended to F^1 by putting $1^\circ = 1$. From Corollaries 2.2 and 2.4, Lemma 2.3 and their duals, we conclude the following.

Theorem 2.5

For $v, w \in (X \cup X^\circ)^+$ we have: if $v = w$ in F, then

$$c(v) = c(w), \qquad v_L \equiv w_L, \qquad v_R \equiv w_R,$$

and

$$\begin{aligned} l(v) &= l(w) \text{ in } F^1 \text{ if } v_L \equiv w_L \not\equiv 1, \\ r(v) &= r(w) \text{ in } F^1 \text{ if } v_R \equiv w_R \not\equiv 1. \end{aligned}$$

Conversely, if $c(v) = c(w)$, then $v = w$ in F if either one of the following occur:

1. (i) $v_L \equiv w_L \not\equiv 1, v_R \equiv w_R \not\equiv 1, l(v) = l(w)$ in F^1 and $r(v) = r(w)$ in F^1,

   (ii) $v_L \equiv w_L \equiv 1, v_R \equiv w_R \not\equiv 1$, and $r(v) = r(w)$ in F^1,

   (iii) $v_L \equiv w_L \not\equiv 1, v_R \equiv w_R \equiv 1$, and $l(v) = l(w)$ in F^1,

   (iv) $v_L \equiv w_L \equiv 1 \equiv v_R \equiv w_R$.

□

We remark here that the task for testing whether $v = w$ in F is, by Theorem 2.5, reduced to a similar task for words of smaller content. Proceeding inductively, Theorem 2.5 thus allows us to verify whether $v = w$ in F in a finite number of steps. We shall be more explicit in the following.

Let $\{l, r\}^*$ be the free monoid on the set containing the symbols l and r, and again let 1 stand for the identity element of $\{l, r\}^*$. For $k \in \{l, r\}^*$ and $w \in (X \cup X^\circ)^*$, we define $k(w)$ inductively by:

$1(w) = w$

and for $k = k_1 \cdots k_n$ with $k_1, \ldots, k_n \in \{l, r\}, n \geq 1$,

$k(w) = k_1(k_2 \ldots k_n(w))$.

The following follows from Theorem 2.5 using induction. For $k \in \{l, r\}^*$, we let $|k|$ be the *length* of k, and, for a set A, we let $|A|$ be the *cardinality* of A.

Theorem 2.6

For $v, w \in (X \cup X^\circ)^+$ we have that $v = w$ in F if and only if $c(v) = c(w)$ and for any $k \in \{l, r\}^*$ with $|k| \leq |c(v)| = |c(w)|$,

$$\begin{aligned} c(k(v)) &= c(k(w)), \\ k(v)_L &\equiv k(w)_L \text{ and } k(v)_R \equiv k(w)_R. \end{aligned}$$

□

From Theorem 2.6 and the solution of the word problem for the free band $F_{\mathbf{B}}(X)$ on the set X (see [2] or Section 4.5 in [3]) it follows that for $v, w \in X^+$, v and w represent the same element of the free band $F_{\mathbf{B}}(X)$ if and only if $v = w$ in F. We can state this as in the following corollary.

Corollary 2.7

Let X be a nonempty set. Then the mapping $X \mapsto X \cup X^\circ$, $x \mapsto x$, can be extended to an embedding of the free band $F_{\mathbf{B}}(X)$ as a subband of $F = F_{\mathbf{B}^\circ}(X)$.

In other words, any free band can be isomorphically embedded into a band which has a semilattice transversal. Therefore, the free objects in the quasivariety consisting of the bands which can be embedded into some band which has a semilattice transversal, are the familiar free bands.

Example 1

The $\mathcal{D}$-classes of $F = F_{\mathbf{B}^\circ}(X)$ are much larger than the corresponding $\mathcal{D}$-classes of the free band $F_{\mathbf{B}}(X)$. Thus, if x and y are distinct elements of X, then the $\mathcal{L}$-class of $x^\circ y^\circ$ in F consists of the 11 elements $x^\circ y^\circ$, $xx^\circ y^\circ$, $y^\circ xx^\circ y^\circ$, $y^\circ yxx^\circ y^\circ$, $yx^\circ y^\circ$, $x^\circ yx^\circ y^\circ$, $x^\circ xyx^\circ y^\circ$, $xyx^\circ y^\circ$, $xx^\circ yx^\circ y^\circ$, $yxx^\circ y^\circ$, and $yy^\circ xx^\circ y^\circ$. Therefore the $\mathcal{D}$-class of xy in F contains exactly 121 elements. The $\mathcal{D}$-class xy in the free band $F_{\mathbf{B}}(X)$ contains only the 4 elements xy, xyx, yx, and yxy.

If $X = \{x, y\}$ then the 11 elements listed above, together with x°, xx°, y°, and yy° form the left regular band which is a transversal of the $\mathcal{R}$-classes of $F_{\mathbf{B}^\circ}(X)$. The 6-element free band $F_{\mathbf{B}}(X)$ is a normal band, whereas the 129-element free split band $F_{\mathbf{B}^\circ}(X)$ is not: $x^\circ F_{\mathbf{B}^\circ}(X) x^\circ$ is not a semilattice, but a 9-element (square) rectangular band with an identity adjoined. If ρ is the smallest fully invariant congruence on $F_{\mathbf{B}^\circ}(X)$ which identifies these 9 elements, then ρ separates the elements of $F_{\mathbf{B}}(X)$: the free normal band $F_{\mathbf{B}}(X)$ is embeddable into the split band $F_{\mathbf{B}^\circ}(X)/\rho$, which is a normal band.

3 Normal forms

We continue to use the notation introduced in the first few paragraphs of the previous section. Thus β is the congruence relation on $(X \cup X^\circ)^+$ generated by the pairs in (2), and as we have seen $(X \cup X^\circ)^+/\beta$ yields the free split band $F_{\mathbf{B}^\circ}(X)$ on the set X. We would like to find a subset of $(X \cup X^\circ)^+$ which is a cross section of the β-classes. The elements of this subset are then called *of normal form*, and every $w \in (X \cup X^\circ)^+$ is β related to a unique element of normal form, called the *normal form of w*. We would like to do all this in such a way that, given any $w \in (X \cup X^\circ)^+$, there exists an algorithm which produces the normal form of w. After this is accomplished, we have a concrete model of $F_{\mathbf{B}^\circ}(X)$ at hand.

In order to be able to find a normal form for the elements of $F_{\mathbf{B}^\circ}(X)$, we shall assume that there exists a total order on the set X of generators. For any finite nonempty set C of X, say $C = \{x_1, \dots, x_n\}$ with $x_1 < x_2 \dots < x_n$ for the total order, we let C° be the uniquely defined word $x_1^\circ \cdots x_n^\circ$ of $(X^\circ)^+$. In particular, if $w \in (X \cup X^\circ)^+$, then $c(w)^\circ \in (X^\circ)^+$ is uniquely defined and $c(c(w)^\circ) = c(w)$.

From the results of the previous section, the following holds.

Lemma 3.1

For any $w \in (X \cup X^\circ)^+, w = l(w)\bar{l}(w)c(w)^\circ\bar{r}(w)r(w)$.

It follows that we can, in a unique way, parse every $w \in (X \cup X^\circ)^+$ following the rule given by Lemma 3.1. We may visualize our parsing of w by considering a tree with a root labeled w, and where every node labeled $v \in (X \cup X^\circ)^+$ is expanded as in Fig. 1.

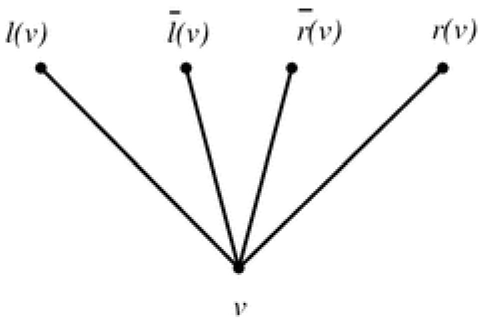

Fig. 1 Expansion of node v

We shall, however, not consider such a further expansion if $v \in X$, $v \equiv 1$ or $v \in (X^\circ)^+$. In the latter case we rewrite v as $c(v)^\circ$. When parsing any $w \in (X \cup X^\circ)^+$ we thus obtain a labeled tree whose leaves are labeled with either 1, or a letter of X, or a word of $(X^\circ)^+$.

By way of example, we give the parsing tree for the word $w \equiv yx^\circ yq^\circ xs^\circ yzyt^\circ z^\circ s$ with $q < s < t < x < y < z$ in X: see Fig. 2. The corresponding normal form for w may then be calculated, using Lemma 3.1, as

$$\begin{aligned} w = \quad & [yx^\circ yq^\circ xs^\circ y] \cdot z \cdot (q^\circ s^\circ t^\circ x^\circ y^\circ z^\circ) \cdot x \cdot [s^\circ yzyt^\circ z^\circ s] \\ = \quad & 1 \cdot y \cdot (q^\circ s^\circ x^\circ y^\circ) \cdot x \cdot [s^\circ y] \cdot z \cdot (q^\circ s^\circ t^\circ x^\circ y^\circ z^\circ) \cdot x \cdot [s^\circ y] \cdot \\ & z \cdot (s^\circ t^\circ y^\circ z^\circ) \cdot y \cdot [t^\circ z^\circ s] \\ = \quad & 1 \cdot y \cdot (q^\circ s^\circ x^\circ y^\circ) \cdot x \cdot s^\circ \cdot y \cdot (s^\circ y^\circ) \cdot y \cdot 1 \cdot z \cdot (q^\circ s^\circ t^\circ x^\circ y^\circ z^\circ) \cdot x \cdot \\ & s^\circ \cdot y \cdot (s^\circ y^\circ) \cdot y \cdot 1 \cdot z \cdot (s^\circ t^\circ y^\circ z^\circ) \cdot y \cdot (t^\circ z^\circ) \cdot s \cdot (s^\circ t^\circ z^\circ) \cdot s \cdot 1. \end{aligned}$$

Here the square brackets indicate that further expansion is required, and dots suggest that some branching in the parsing tree is involved.

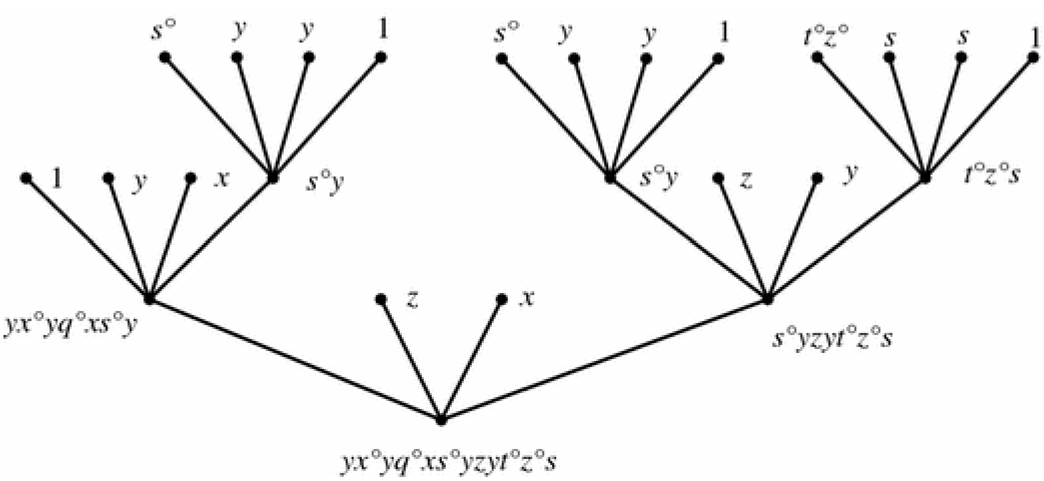

Fig. 2 Parsing of the word w

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